Physics 252 Final Examination

Friday, May 12, 2005 (10:30am)

The test consists of 8 problems, choose 5 problems out of the 8. Show **ALL** steps and be very neat or you **will** lose points. You have 120 minutes to complete the test.

Problem 1

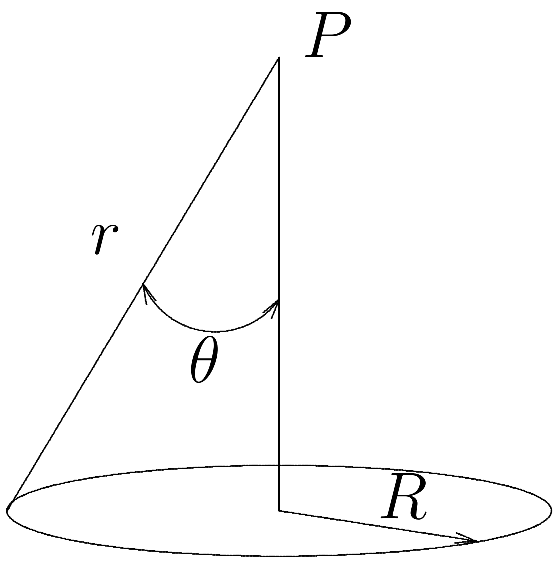

Figure 1: A circular ring of positive linear charge density λ Coulombs per meter.

Figure 1 shows a circular ring of radius R, positive linear charge density λ and total charge q. Use Coulombs law for electric fields: $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ to find the net electric field at the point P which is at a position z above the center of the ring in terms of q, z and R.

Problem 1 – Solution

Let ds be a small element of arc length of the ring. Its charge will be $dq = \lambda ds$. The field from it will then be

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{r^2}$$

but $r^2 = R^2 + z^2$ and so

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{R^2 + z^2}$$

The x and y contributions will cancel at P due to symmetry. The Z component will be $|dE|\cos\theta$ and since $\cos\theta = z/r$ we have

$$E = \int |dE|\cos\theta = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi R}\frac{\lambda z ds}{(R^2+z^2)^{\frac{3}{2}}}$$

Since $q = \lambda(2\pi R)$ and everything is constant in the integrand and so our result is

$$E = \frac{1}{4\pi\epsilon_0} \frac{qz}{(R^2 + z^2)^{\frac{3}{2}}}$$

Problem 2

A non-conducting sphere has radius R and a uniformly distributed charge of q throughout its volume. The electric potential at the sphere's center will be defined as $V_0 = 0$. You will need to use Gauss' law to find an expression for the electric field at an arbitrary point inside the sphere and then use it to find:

(a) the potential at a radial distance r inside the sphere.

(b) the potential at the surface $r = R$ of the sphere.

Problem 2 – Solution

In order to find the potential, we first need the electric field. Using a spherical Gaussian surface at radius r inside the conducting sphere we have:

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\mathsf{enc}} = \left(\frac{q}{\frac{4}{3}\pi R^3} \right) \frac{4}{3}\pi r^3$$

Since $\vec{E}$ is in the radial direction and is constant at any given radius (it only changes if you change the radius) it will come out of the integral. We are left with

$$\epsilon_0 E \oint dA = E(4\pi r^2) = \frac{qr^3}{R^3}$$

and thus $E = \frac{qr}{4\pi\epsilon_0 R^3}$.

Now with this expression for E we can find:

(a) The potential is given by

$$\begin{aligned} V &= -\int_0^r \frac{qr}{4\pi\epsilon_0 R^3} dr \\ &= -\frac{q}{4\pi\epsilon_0 R^3} \int_0^r r dr \\ &= -\frac{qr^2}{8\pi\epsilon_0 R^3} \end{aligned}$$

(b) At $r = R$ the above expression reduces to

$$V = -\frac{q}{8\pi\epsilon_0 R}$$

Problem 3

A cylindrical capacitor of inner radius a and outer radius b has a charge q (negative q on the outer surface and positive q on the inner surface). Use $q = CV$ to find the capacitance. Hint: Use Gauss' law to find E(r) and then use a path integral from the negatively charged surface to the positively charged one to find V(r) from E(r).

Problem 3 – Solution

Gauss' law for our cylindrical Gaussian surface gives

$$q = \epsilon_0 EA = \epsilon_0 E(2\pi rL)$$

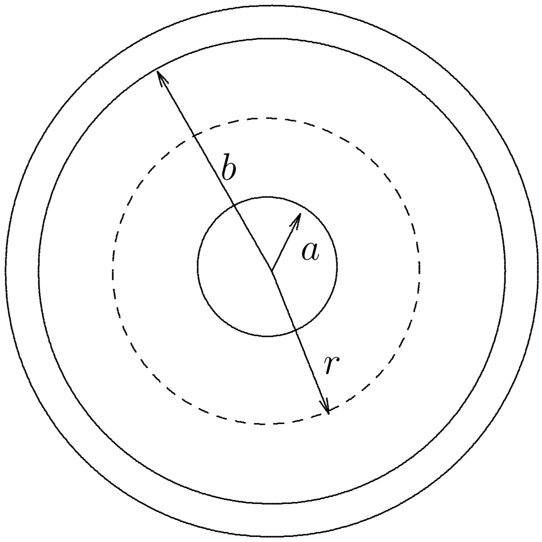

Figure 2: A cross section of a long cylindrical capacitor. The dashed line shows a cylindrical Gaussian surface of radius r.

The potential is then

$$V = \int_{-}^{+} E ds = \frac{q}{2\pi\epsilon_0 L}\int_a^b \frac{dr}{r}$$

which is $V = \frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$. So the capacitance is thus

$$C = \frac{q}{V} = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$

Problem 4

Ampere's law says that if you integrate around a closed loop, $\oint \vec{B}\cdot d\vec{s} = \mu_0 i$ where i is the current enclosed in the loop and $\vec{B}$ is the magnetic field along the loop. Using Ampere's law we can show that the artists drawing of the magnetic field between a magnetic north pole and a magnetic south pole as shown in Figure 3 is not correct. Show this and then explain how the diagram should be modified in order to be more accurate. Hint: Make an rectangular Amperian loop at the edge of the magnetic field, with one side inside the field and the other outside of it.

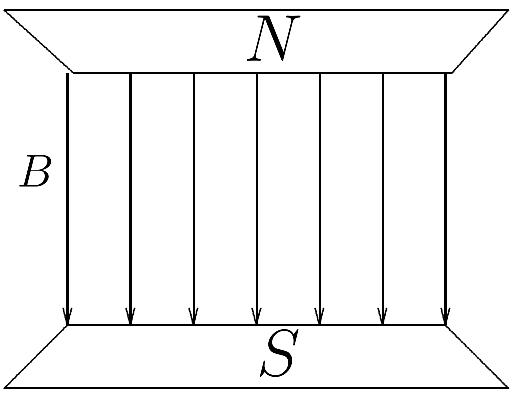

Figure 3: An artists rendition of the magnetic field between two magnets. The problem is to explain what is wrong with this diagram by using Ampere's law.

Problem 4 – Solution

If we construct an Amperian loop with one side of length L in the field at the edge of the diagram and the other outside of the field, Ampere's law gives

$$\mu_0 i = \oint \vec{B} \cdot d\vec{s} = BL + 0 + 0 + 0$$

where the outside contribution is zero since there is no field there. So $i = \frac{BL}{\mu_0}$. But there is no current inside the loop! So the diagram cannot be correct. The solution is that there should be some magnetic field lines drawn at the edge. These are the so called 'edge effects'

Problem 5

Faradays law of induction states that the line integral of the eletric field over a closed path is equal to the negative rate of change of magnetic flux over time: $\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$.

Suppose we have a long solenoid of radius R with n turns per unit length. The current in the wire varies sinosoidally as $I = I_m \cos \omega t$ where I_m is the maximum current and ω is the angular frequency of the alternating current source.

Determine the magnitude of the induced electric field outside the solenoid at a distance $r > R$ from its long central axis.

Problem 5 – Solution

The magnetic flux is given by $\Phi = \int \vec{B} \cdot d\vec{A}$. The magnetic field can be found from Ampere's law:

$$\mu_0 n L I = LB \Rightarrow B = \mu_0 n I$$

and so

$$\Phi = BA = \mu_0 n I (\pi R^2)$$

and thus Faraday's law gives

$$\begin{aligned} -\frac{d\Phi}{dt} &= -\mu_0 n \pi R^2 \frac{dI}{dt} = \mu_0 n \pi R^2 I_m \omega \sin \omega t \\ \Rightarrow E(2\pi r) &= \mu_0 n \pi R^2 I_m \omega \sin \omega t \\ E &= \frac{\mu_0 n I_m \omega R^2}{2r} \sin \omega t \end{aligned}$$

Problem 6

A spear fisherman spots a fish of length L under water (index of refraction n). The center of the fish is at a position x meters from the edge of the pond and at a depth of d meters from the surface of the pond as shown. The real fish is directly below the image that that the fisherman sees. What is the actual depth of the real fish in terms of L, n and d? Note: The index of refraction of air can be taken as $n = 1$. Comment: Use the ray shown on the diagram (given during the test).

Problem 6 – Solution

Snell's law gives $n \sin \theta_1 = \sin \theta_2$ and using trigonometry we find that

$$\sin \theta_1 = L/\sqrt{L^2 + D^2}$$

where D is the real depth of the fish. Also

$$\sin \theta_2 = L/\sqrt{L^2 + d^2}$$

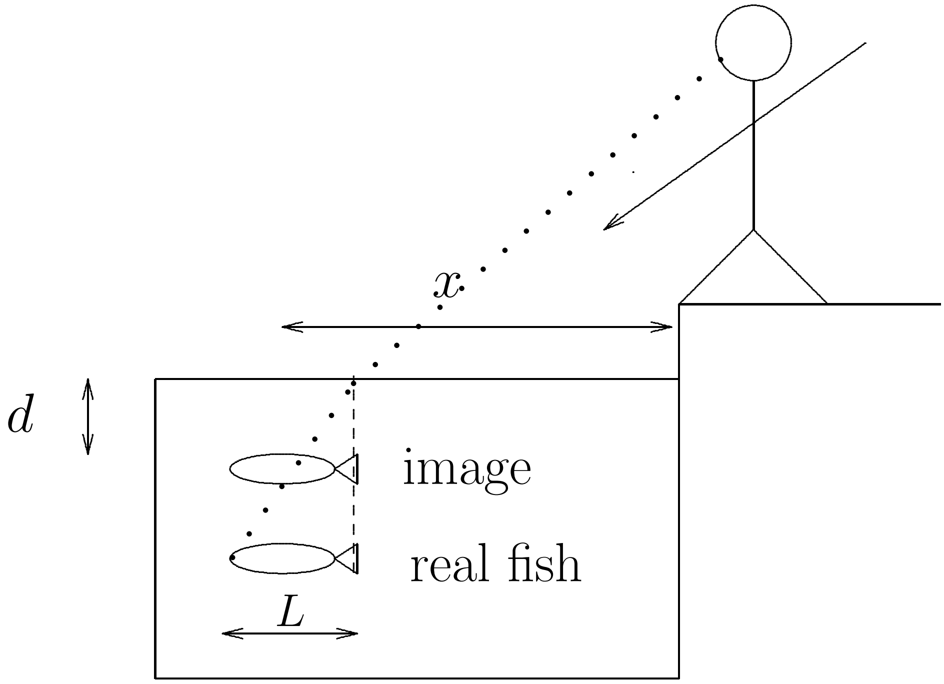

Figure 4: A spear fisherman sees a fish.

and so Snell's law gives

$$\begin{aligned} nL/\sqrt{L^2+D^2} &= L/\sqrt{L^2+d^2} \\ n^2/(L^2+D^2) &= 1/(L^2+d^2) \\ D^2 &= n^2(L^2+d^2)-L^2 \\ D &= \sqrt{n^2(L^2+d^2)-L^2} \end{aligned} \tag{1}$$

Problem 7

A beetle is placed a distance of x cm in front of a two converging lens system. The lens (lens 1) closer to the beetle has a focal length of $f_1 = x/2$ cm and the other lens (lens 2) has a focal length of $2f_1/3$ cm. The distance between the two lenses is $d = 4f_1$. Draw a lens diagram. What are

(a) The final image distance (in terms of x).

(b) The final image orientation inverted or not inverted.

(c) The final image type, real or virtual.

(d) The overall lateral magnification.

Problem 7 – Solution

(a) To do this we first find the image through the first lens using the thin lens formula:

$$\begin{aligned} \frac{1}{f} &= \frac{1}{i}+\frac{1}{P} \\ \frac{2}{x} &= \frac{1}{i}+\frac{1}{x} \\ i &= x \end{aligned} \tag{2}$$

and now this image becomes the object for the second lens Notice that the image is at a distance of x from the second lens.

$$\begin{aligned} \frac{3}{x} &= \frac{1}{i} + \frac{1}{x} \\ \frac{2}{x} &= \frac{1}{i} \\ i &= \frac{x}{2} \end{aligned} \tag{3}$$

(b) The image is not inverted. It is upright.

(c) The image is real.

(d) The overall lateral magnification is given by

$$\begin{aligned} m_1 &= -\frac{i_1}{p_1} = -\frac{x}{x} = -1 \\ m_2 &= -\frac{i_2}{p_2} = -\frac{x}{2x} = -\frac{1}{2} \end{aligned}$$

and so the total is

$$m = m_1 m_2 = \frac{1}{2}$$

so upright and half as big.

Problem 8

(a) The quantity τ defined by $c^2\tau^2 = c^2t^2 - \vec{r}^2$ is called the 'proper time'. Show that the proper time is invariant under a Lorentz transformation in the y-direction. Is it invariant under a Lorentz transformation in the x or the z direction?

(b) Given the fact that $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$ show that $E^2 - \vec{p}^2c^2 = m^2c^4$. Note that this equation for relativistic energy reduces to the famous $E = mc^2$ if the particle is not moving (i.e. when the momentum $\vec{p}$ is zero).

Problem 8 – Solution

(a) A Lorentz transformation in the y direction is given by:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= x \\ y' &= \gamma\left(y - vt\right) \\ z' &= z \end{aligned}$$

So the proper time becomes

$$\begin{aligned} c^2\tau^2 &\to c^2(t')^2 - (\vec{r}')^2 \\ &= c^2(\gamma\left(t - \frac{vy}{c^2}\right))^2 - x^2 - (\gamma(y - vt))^2 - z^2 \\ &= \gamma^2\left(c^2t^2 - 2vyt + \frac{v^2y^2}{c^2} - y^2 + 2yvt - v^2t^2\right) - x^2 - z^2 \\ &= \gamma^2\left((c^2 - v^2)t^2 - (1 - \frac{v^2}{c^2})y^2\right) - x^2 - z^2 \\ &= c^2t^2 - y^2 - x^2 - z^2 \\ &= c^2t^2 - \vec{r}^2 \end{aligned} \tag{4}$$

In the x or the z direction the exact same thing happens.

(b) $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$ so

$$\begin{aligned} E^2 - \vec{p}^2 c^2 &= m^2 c^4 \\ (\gamma mc^2)^2 - (\gamma m\vec{v})^2 c^2 &= m^2 c^4 \\ \gamma^2 m^2 c^4 - \gamma^2 m^2 \vec{v}^2 c^2 &= m^2 c^4 \\ \gamma^2 c^2 - \gamma^2 \vec{v}^2 &= c^2 \\ \gamma^2 (c^2 - \vec{v}^2) &= c^2 \\ \gamma^2 &= \frac{1}{1 - \frac{\vec{v}^2}{c^2}} \end{aligned} \tag{5}$$

which shows that they are equivalent.

The End
Have a nice summer!